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LETTER TO THE EDITOR

Stationary distributions of stochastic processes described by a linear neutral delay differential equation

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Abstract
Stationary distributions of processes are derived that involve a time delay and are defined by a linear stochastic neutral delay differential equation. The distributions are Gaussian distributions. The variances of the Gaussian distributions are either monotonically increasing or decreasing functions of the time delays. The variances become infinite when fixed points of corresponding deterministic processes become unstable.

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Neutral delay differential equations have been used to describe various processes in physics and engineering sciences [1, 2]. For example, transmission lines involving nonlinear boundary conditions [1], cell growth dynamics [3], propagating pulses in cardiac tissue [4] and drill-string vibrations [5] have been described by means of neutral delay differential equations. From a mathematical point of view, neutral delay differential equations are regarded as a particular class of functional differential equations [1, 6]. While for deterministic processes described by neutral (and functional) delay differential equations, several helpful results have been derived [1, 2, 6–11] and for stochastic processes by retarded (but not neutral) delay differential equations stationary distributions have been obtained [12–19]; for stochastic processes described by neutral delay differential equations comparatively little is known. However, stochastic processes play important roles in physics and various other disciplines. Therefore, there is a need to examine not only deterministic but stochastic processes in the framework of neutral delay differential equations. Since at present such a stochastic approach is in its infancy, in what follows, we will examine one of the most simple but non-trivial neutral delay differential equations at hand: a linear one which is of first order and involves a point delay.

Let $X(t)$ denote a random variable defined on the real line. Let $X(t)$ describe a stochastic processes given by

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = -aX(t) - b\frac{\mathrm{d}}{\mathrm{d}t}X(t-\tau) + \sqrt{Q}\Gamma(t) \tag{1}$$

for $t \geqslant 0$. Here, $\Gamma(t)$ is a Langevin force with Gaussian characteristic functional [20, 21] and $\langle\Gamma\rangle = 0$, $\langle\Gamma(t)\Gamma(t')\rangle = \delta(t-t')$, where $\delta(\cdot)$ denotes the delta function and $\langle\cdot\rangle$ denotes an ensemble averaging. The expression $\sqrt{Q}\Gamma(t)$ describes a fluctuating force or a noise source. Accordingly, $Q \geqslant 0$ will be referred to as the noise amplitude. In addition, we assume that for $z \in [-\tau, 0]$ we have $\mathrm{d}X(z)/\mathrm{d}z = \varphi(z)$ and $X(0) = x_0$, which fixes the initial conditions. For $b = 0$ equation (1) describes an Ornstein–Uhlenbeck process [20]. For $b \neq 0$ equation (1) describes a non-Markovian stochastic process with memory.

Let us briefly recall what is known for the deterministic dynamics related to equation (1). For $Q = 0$ equation (1) has the fixed point $X_{\mathrm{st}} = 0$. For $|b| < 1$ the fixed point is asymptotically stable [1, 2], whereas for $|b| > 1$ it is unstable [1]. In the context of linear functional differential equations, one can conclude from the stability of deterministic processes to the existence of the stationary distributions of the corresponding stochastic process [12, 16]. If fixed points are asymptotically stable (unstable), then stationary distributions exist (do not exist). Consequently, for $Q > 0$ the stochastic process given by equation (1) has a stationary distribution for $|b| < 1$ and does not exhibit a stationary distribution for $|b| > 1$.

In what follows, let us consider the case $|b| < 1$. From equation (1) it is clear that in the stationary case the mean value vanishes. Moreover, since we are dealing with a linear evolution equation and a fluctuating force $\Gamma(t)$ that has a Gaussian characteristic functional, we conclude that the stationary distribution $P_{\mathrm{st}}(x) = \langle\delta(x - X(t))\rangle_{\mathrm{st}}$ is a Gaussian distribution. That is, we have

$$P_{\mathrm{st}}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}, \tag{2}$$

where σ^2 denotes the variance of $X(t)$.

In order to compute the variance, we use the autocorrelation method developed for stochastic delay differential equations of the retarded type [12, 13, 22–24]. Let $C(z) = \langle X(t)X(t+z)\rangle_{\mathrm{st}}$ denote the autocorrelation function of $X(t)$ in the stationary case. Then the variance is given by $\sigma^2 = C(0)$. Using equation (1), the symmetry $C(z) = C(-z)$, and the fact that $\langle x(t)\Gamma(t+z)\rangle = 0$, for $z \in (0, \tau)$ (hint: use method of steps as in [23]), we obtain

$$\frac{\mathrm{d}C(z)}{\mathrm{d}z} = -a\frac{\mathrm{d}C(z)}{\mathrm{d}z} - b\frac{\mathrm{d}C(\tau - z)}{\mathrm{d}z} \tag{3}$$

for $z > 0$. Substituting

$$C(z) = A\sinh\left(\omega\left(z - \frac{\tau}{2}\right)\right) + B\cosh\left(\omega\left(z - \frac{\tau}{2}\right)\right) \tag{4}$$

for $z > 0$ into equation (3), we obtain $\omega = a/\sqrt{1-b^2}$ and $A = -\sqrt{(1+b)/(1-b)}B$. Next, we need to determine the parameter B. Multiplying equation (1) with $X(t)$ and averaging and multiplying equation (1) with $bX(t-\tau)$ and averaging and adding both results, we find

$$a[C(0) + bC(\tau)] = \sqrt{Q}\langle(X(t) + bX(t-\tau))\Gamma(t)\rangle_{\mathrm{st}}. \tag{5}$$

Exploiting Novikov's theorem, which states that $\langle R(\Gamma)\Gamma\rangle = \langle\delta R/\delta\Gamma\rangle$ [21, 25], we find $\sqrt{Q}\langle(X(t) + bX(t-\tau))\Gamma(t)\rangle = Q/2$. Consequently, we have the constraint

$$a[C(0) + bC(\tau)] = \frac{Q}{2}. \tag{6}$$

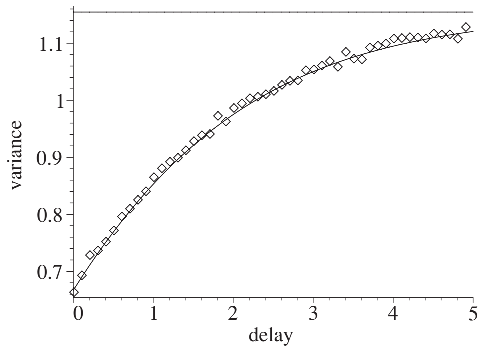

Figure 1. Variance σ^2 of the stochastic process defined by equation (1) as a function of the time delay τ. Solid line: analytical result (8). Diamonds: numerical results obtained by solving equation (1) numerically using a modification of the Euler forward discretization scheme for ordinary Langevin equations (single time step $\Delta t = 0.01$; number of realizations $N = 10^7$; Gaussian random numbers via Box–Muller algorithm) [20]. Solid horizontal line: asymptotic value of σ^2 computed from $\sigma^2(\infty) = Q/[2a\sqrt{1-b^2}]$. Parameters: $a = 0.5, b = 0.5, Q = 1$ ($\Rightarrow \sigma^2(\infty) \approx 1.15$).

Substituting equation (4) into equation (6) together with $A = -\sqrt{(1+b)/(1-b)}B$, we can determine B. Taking the symmetry $C(z) = C(-z)$ into account, we finally get

$$C(z) = \frac{Q}{2a(1+b)} \left[\frac{\cosh\left(\omega\left(|z| - \frac{\tau}{2}\right)\right) - \sqrt{\frac{1+b}{1-b}} \sinh\left(\omega\left(|z| - \frac{\tau}{2}\right)\right)}{\cosh\left(\frac{\omega\tau}{2}\right) + \sqrt{\frac{1-b}{1+b}} \sinh\left(\frac{\omega\tau}{2}\right)} \right]. \tag{7}$$

In particular, the variance $\sigma^2 = C(0)$ reads

$$\sigma^2 = \frac{Q}{2a(1+b)} \left[\frac{\cosh\left(\frac{\omega\tau}{2}\right) + \sqrt{\frac{1+b}{1-b}} \sinh\left(\frac{\omega\tau}{2}\right)}{\cosh\left(\frac{\omega\tau}{2}\right) + \sqrt{\frac{1-b}{1+b}} \sinh\left(\frac{\omega\tau}{2}\right)} \right]. \tag{8}$$

Let us consider two special cases. For $\tau = 0$ we have $\sigma^2 = Q/[2a(1+b)]$, which is the variance of the corresponding Ornstein–Uhlenbeck process. For $\tau \to \infty$ we obtain $\sigma^2 = Q/[2a\sqrt{1-b^2}]$. How does σ^2 depend on τ in between these two extremes? Differentiating equation (8) with respect to τ, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma^2 = \frac{bQ}{2(1+b)\sqrt{1-b^2}} \left[\cosh\left(\frac{\omega\tau}{2}\right) + \sqrt{\frac{1-b}{1+b}} \sinh\left(\frac{\omega\tau}{2}\right) \right]^{-2}. \tag{9}$$

Consequently, we have $\mathrm{d}\sigma^2/\mathrm{d}\tau > 0$ for $b > 0$ and $\mathrm{d}\sigma^2/\mathrm{d}\tau < 0$ for $b < 0$. In other words, for $b > 0$ the variance increases monotonically from $\sigma^2 = Q/[2a(1+b)]$ at $\tau = 0$ to $\sigma^2 = Q/[2a\sqrt{1-b^2}]$ for $\tau \to \infty$. For $b < 0$ the variance decreases monotonically from $\sigma^2 = Q/[2a(1-|b|)]$ at $\tau = 0$ to $\sigma^2 = Q/[2a\sqrt{1-b^2}]$ for $\tau \to \infty$. Figures 1 and 2 illustrate the behaviour of σ^2 as a function of τ for $b > 0$ and $b < 0$. Both the analytical solution (8) and results obtained from computer simulations are shown.

In closing our considerations on the stochastic neutral delay differential equation (1), let us have a look at the situation for $b = \pm 1$. In the deterministic case (i.e. for $Q = 0$), the real parts of the eigenvalues λ of the characteristic equation related to equation (1) become arbitrarily close to zero [1]. That is, there does not exist an $\epsilon > 0$ such that $\mathrm{Re}\{\lambda\} < -\epsilon$ holds for all eigenvalues. Therefore, the dynamics effectively behaves like a neutrally stable

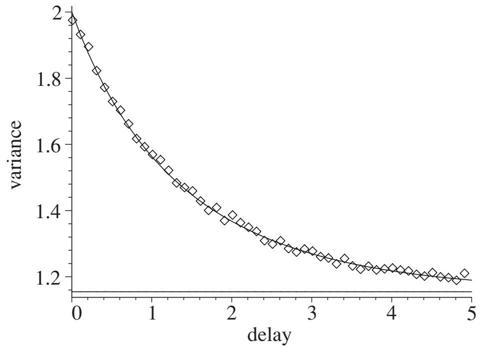

Figure 2. As in figure 1 but for $b = -0.5$.

dynamics characterized by eigenvalues (or Lyapunov exponents) with vanishing real parts. In the stochastic case (i.e. for $Q > 0$), we can compute from equation (8) the variance σ^2 in the limits $b \to \pm 1$. To this end, we first note that for $b \to \pm 1$ we have $\omega \to \infty$. Next, we note that $\cosh(\omega\tau/2) \to \exp(\omega\tau/2)/2$ and $\sinh(\omega\tau/2) \to \exp(\omega\tau/2)/2$ in the limit $\omega \to \infty$. Substituting these expressions into equation (8), we have $b \to \pm 1 \Rightarrow \sigma^2 \to Q/[2a\sqrt{1-b^2}]$ which implies that $\sigma^2 \to \infty$. That is, the variance tends to infinity at the boundaries of the domain of asymptotic stability. The reason for this divergence is that stochastic processes related to linear deterministic systems with neutrally (but not asymptotically) stable fixed points do not exhibit stationary distributions [12, 16]. In other words, in the limit $b \to \pm 1$ we have $\mathrm{Re}\{\lambda\} \to 0$ and the fluctuations caused by the driving force $\Gamma(t)$ occurring in equation (1) cannot be sufficiently suppressed by the restoring force $f(X) = -aX(t) - b\,\mathrm{d}X(t-\tau)/\mathrm{d}t$.

There are at least two fundamental applications of the linear stochastic model (1). First, it can be used in the context of linearized stochastic time-delayed differential equations of the neutral type. Second, it may be used to study nonlinear neutral stochastic delay differential equations that can be transformed into linear ones. Both applications have been previously discussed for retarded stochastic time-delayed differential equations [16, 17]. Let us illustrate these points by some examples taken from the theory of population dynamics. The evolution of a population is often described by means of a first-order differential equation $\mathrm{d}N/\mathrm{d}t = rN(t)\Phi$, where $N \in [0, \infty)$ is the size of the population, $r \geqslant 0$ is a parameter and Φ is the so-called regulation function [26]. For $\Phi = 1$ we have exponential growth. For $\Phi = \Phi(N)$ we can model saturation effects. As pointed out in [27] the regulation function Φ can also depend on the change of the population given by $\mathrm{d}N/\mathrm{d}t$. Taking a scaling function $s(N)$ into account (i.e. replacing $\mathrm{d}N/\mathrm{d}t$ by the more general expression $\mathrm{d}s(N)/\mathrm{d}t$), we then obtain $\Phi = \Phi(N, \mathrm{d}s(N)/\mathrm{d}t)$. In addition, we may account for time lags, which finally leads to $\Phi = \Phi[N(t-\tau_1), \mathrm{d}s(N(t-\tau_2))/\mathrm{d}t]$. In what follows, we will consider the case $\tau_1 = 0$ and $\tau_2 = \tau$. Supplementing the deterministic evolution equation $\mathrm{d}N/\mathrm{d}t = rN(t)\Phi$ with a fluctuating force [26], we end up with stochastic neutral delay differential equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = rN(t)\Phi\left[N(t), \frac{\mathrm{d}}{\mathrm{d}t}s(N(t-\tau))\right] + g(N)\Gamma(t), \tag{10}$$

where g is a noise amplitude that may depend on the population size. The linearization of equation (10) in the case of a weak additive noise source (e.g. $g(N) = \sqrt{Q}$ with Q small) yields a linear equation of the form (1). For example, as suggested in [27], let us consider a generalized logistic model with $\Phi = 1 - K^{-1}[N + \rho\, \mathrm{d}N(t-\tau)/\mathrm{d}t]$, $K > 0$, and $\rho \in \mathbb{R}$. That is, we are dealing with

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = rN(t)\left[1 - \frac{1}{K}\left(N(t) + \rho\frac{\mathrm{d}}{\mathrm{d}t}N(t-\tau)\right)\right] + \sqrt{Q}\Gamma(t). \tag{11}$$

Linearization at the stationary value $N_{\mathrm{st}} = K$ that we would obtain for $Q = 0$ gives us equation (1) with $X(t) = N(t) - K$, $a = r$ and $b = \rho r$. Requiring $|\rho r| < 0$, the stationary distribution of X is given by equation (2). Consequently, the stationary distribution in terms of N is given by

$$P_{\mathrm{st}}(N) \approx \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{[N-K]^2}{2\sigma^2}\right\} \tag{12}$$

with $\sigma^2 = \sigma^2\,(a = r, b = \rho r)$. Note that in driving equation (12) we assume that the noise amplitude Q is small enough such that the probability to find negative N-values becomes negligible. In line with earlier suggestions on a generalized Gompertz model [16, 17], we consider in our second example the regulator function $\Phi = -[\ln(N/c) + \rho\,\mathrm{d}\ln(N(t-\tau)/c)/\mathrm{d}t]$ with $c > 0$ and $g(N) = \sqrt{Q}N$. Accordingly, we are concerned with the stochastic model

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = -r\left[N(t)\ln\left(\frac{N(t)}{c}\right) + \rho N(t)\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(\frac{N(t-\tau)}{c}\right)\right] + \sqrt{Q}N\Gamma(t), \tag{13}$$

where the noise term is defined by the Stratonovich calculus [28]. By means of the variable transformation $X(t) = \ln(N(t)/c)$, the generalized Gompertz model (13) can be transformed into equation (1) with $a = r$ and $b = \rho r$. Using $P_{\mathrm{st}}(N) = P_{\mathrm{st}}(x)|\mathrm{d}x/\mathrm{d}N|$, for $|\rho r| < 1$, the stationary distribution of equation (13) is given by

$$P_{\mathrm{st}}(N) = \frac{1}{N\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{[\ln(N/c)]^2}{2\sigma^2}\right\} \tag{14}$$

with $\sigma^2 = \sigma^2\,(a = r, b = \rho r)$.

We have derived the stationary distribution of a linear stochastic delay differential equation of neutral type in terms of a Gaussian distribution. In particular, the variance of the Gaussian distribution as a function of the time delay has been determined. The stationary Gaussian distribution exists in the parameter domain for which the fixed point of the associated deterministic system is asymptotically stable. When the fixed point becomes unstable, then the variance tends to infinity and the stationary distribution does no longer exist. Applications in the realm of population dynamics show that to a certain extent the results obtained for the linear model can also be used to address nonlinear stochastic models of the neutral type.

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